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A Modified Fractional Step Method of Keeping a Constant Mass Flow Rate in Fully Developed Channel and Pipe Flows

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The objective of this paper is to present a modified fractional step method of keeping a constant mass flow rate in spatially periodic flows, because original fractional step methods do not precisely keep the mass flow rate constant in time. In the modified method, the mean and fluctuating pseudo-pressure gradients are separately obtained at each time step. This method is successfully applied to channel and pipe flows and shown to be suitable for maintaining a constant mass flow rate in time.

Key Words : Fractional Step Method, Mass Flow Rate, Spatially Periodic Flows, Channel Flow, Pipe Flow

1. Introduction

For a direct or large eddy simulation of a spatially periodic turbulent channel or pipe flow, a periodic boundary condition is used in the streamwise direction instead of an inflow/outflow boundary condition. Under this condition, an external forcing term is required to maintain flow inside a channel or pipe.

There have been two methods of providing the external forcing term: one is fixing the mean pressure gradient in time (Kim *et al.*, 1987; Eggels *et al.*, 1994), and the other is fixing the mass flow rate in time (Choi *et al.*, 1994; Orlandi & Fatica, 1997; Lee *et al.*, 1998). In the latter case, the mean pressure gradient is obtained at each time step by integrating the Navier—Stokes equation in the streamwise direction over a computational domain, so as to maintain a constant mass flow rate. However, the algorithm of keeping a constant mass flow rate is not clearly specified in the literature and is not evidently validated.

Recently, the turbulence control for drag reduction and heat-transfer enhancement becomes an important issue in fluid mechanics, and drag reduction in a turbulent channel or pipe flow is a building-block problem for developing control strategies. In most studies, researchers have imposed a constant mass flow rate and monitored changes in the mean pressure gradient necessary to drive the flow with a fixed mass flow rate. Therefore, it is important to verify that the fractional step methods used in the literature indeed keep a constant mass flow rate in a channel or pipe.

In the present paper, we show that the original fractional step methods do not keep the mass flow rate constant, and thus we suggest a modified fractional step method of keeping the constant mass flow rate in a fully developed channel or pipe flow.

2. Numerical Method

The governing equations for an incompressible flow are

$$\frac{\partial u_i}{\partial t}+\frac{\partial u_j u_i}{\partial x_j}=-\frac{\partial p}{\partial x_i}+\frac{1}{Re}\frac{\partial^2 u_i}{\partial x_j \partial x_j}, \quad (1)$$

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$$\frac{\partial u_i}{\partial x_i}=0, \tag{2}$$

where x_i's are the Cartesian coordinates, u_i's are the corresponding velocity components, p is the pressure, and t is the time. All the variables are non-dimensionalized by a bulk velocity and a length scale, and Re is the Reynolds number defined with these scales. The pressure gradient $-\frac{\partial p}{\partial x_i}$ is split into a mean pressure gradient $-\frac{dP}{dx_1}\delta_{i1}$ and a fluctuating pressure gradient $-\frac{\partial p'}{\partial x_i}$ such that periodic boundary conditions in the streamwise direction can be employed for the fluctuating pressure p'. Here, x_1 is the streamwise direction.

Let us consider a semi-implicit method for the integration of Eqs. (1) and (2), which has been widely used in the literature (Kim & Moin, 1985; Le & Moin, 1991; Akselvoll & Moin, 1996). Let A_i and B_i represent the terms treated explicitly (e.g., third-order Runge—Kutta) and implicitly (e.g., Crank-Nicolson), respectively. A_i and B_i are defined differently for channel and pipe geometries (see Sec. 3). Then the time-advancement scheme for Eqs. (1) and (2) is expressed as:

$$\frac{u_i^k-u_i^{k-1}}{\Delta t}=\beta_k(B_i^k+B_i^{k-1}) + \gamma_k A_i^{k-1}+\zeta_k A_i^{k-2}-2\beta_k\frac{\partial p^k}{\partial x_i}, \tag{3}$$

$$\frac{\partial u_i^k}{\partial x_i}=0, \tag{4}$$

where $k(k=1, 2, 3)$ denotes the Runge—Kutta sub-step such that $u^0=u^n$ and $u^3=u^{n+1}$, and n denotes the time step. The coefficients, β_k, γ_k and ζ_k, are

$$\beta_1=\frac{4}{15},\ \beta_2=\frac{1}{15},\ \beta_3=\frac{1}{6},\quad \gamma_1=\frac{8}{15},\ \gamma_2=\frac{5}{12},\ \gamma_3=\frac{3}{4},\quad \zeta_1=0,\ \zeta_2=-\frac{17}{60},\ \zeta_3=-\frac{5}{12}. \tag{5}$$

Equations (3) and (4) have been solved using the following original fractional step method:

$$\hat{u}_i-\Delta t\beta_k\hat{B}_i=u_i^{k-1}+\Delta t(\beta_k B_i^{k-1} + \gamma_k A_i^{k-1}+\zeta_k A_i^{k-2})-2\beta_k\Delta t\frac{\partial p^{k-1}}{\partial x_i}, \tag{6}$$

$$\frac{\partial^2\phi}{\partial x_i\partial x_i}=\frac{1}{2\beta_k\Delta t}\frac{\partial\hat{u}_i}{\partial x_i}, \tag{7}$$

$$u_i^k=\hat{u}_i-2\beta_k\Delta t\frac{\partial\phi}{\partial x_i}, \tag{8}$$

where $\hat{u}_i$ is the intermediate velocity, and ϕ is a scalar (pseudo-pressure) to be determined. By substituting Eq. (8) into (6) and comparing with Eq. (3), one can easily find

$$p^k=p^{k-1}+\phi+O(\Delta t^2). \tag{9}$$

In order to show the weakness of this fractional step method in keeping a constant mass flow rate for spatially periodic flows, we consider a very simple problem: temporal development of the velocity profile in a channel. The initial velocity is uniform across the channel, and we want to keep the mass flow rate constant as the flow evolves in time. In this specific problem, let A_i and B_i represent the convection and diffusion terms, respectively. The streamwise and wall-normal directions are x_1 and x_2, respectively, and u_1 and u_2 are the corresponding velocity components. Because $u_2=0$ and $\frac{\partial}{\partial x_1}(\cdot)=0$ for this specific flow, all the convection terms are zero (i.e. $A_i=0$) and divergence of the intermediate velocity is zero from Eq. (6). In solving Eq. (7), previous studies have used a periodic boundary condition for ϕ, i.e. $\frac{\partial\phi}{\partial x_1}=0$, so that Eq. (7) becomes $\frac{\partial^2\phi}{\partial x_2^2}=0$. Because $u_2=0$ and $\frac{\partial\phi}{\partial x_2}=0$ (Eq. 8) on the wall, $\phi=constant$. Then from Eq. (8), $u_i^k=\hat{u}_i$. Hence, the mass flow rate is not changed at the last step (Eq. (8)). Integration of Eq. (6) in the streamwise direction gives

$$\int\hat{u}_1 dV-\int u_1^{k-1}dV=\Delta t\beta_k\Big(\int\hat{B}_1 dV + \int B_1^{k-1}dV-2\int\frac{\partial p^{k-1}}{\partial x_1}dV\Big), \tag{10}$$

where V is the volume of the computational domain. The mean pressure gradient for a constant mass flow rate is obtained by integrating Eq. (1): for $\frac{\partial}{\partial t}\int u_1 dV=0$, $\frac{dP}{dx_1}=\frac{1}{V}\int B_1 dV$. Then, considering that $\int \hat{B}_1 dV = \int B_1^k dV = V\frac{dP^k}{dx_1}$ and $\int B_1^{k-1} dV = V\frac{dP^{k-1}}{dx_1}$, the right hand side of Eq. (10) becomes $\Delta t \beta_k V\left(\frac{dP^k}{dx_1} - \frac{dP^{k-1}}{dx_1}\right)$, which is not necessarily zero. Therefore $\int u_1^k dV \neq \int u_1^{k-1} dV$.

This inequality results from the treatment of ϕ. From Eq. (9) we can easily find that the change of the mean pressure gradient is caused by the non-zero mean gradient of ϕ. Therefore, $\frac{\partial \phi}{\partial x_i}$ should be split into a mean value $\frac{d\Phi}{dx_1}\delta_{i1}$ and a fluctuating one $\frac{\partial \phi'}{\partial x_i}$. Then, Eqs. (7) and (8) are modified as follows:

$$\frac{\partial^2 \phi'}{\partial x_i \partial x_i} = \frac{1}{2\beta_k \Delta t}\frac{\partial \hat{u}_i}{\partial x_i}, \tag{11}$$

$$u_i^k = \hat{u}_i - 2\beta_k \Delta t\left(\frac{d\Phi}{dx_1}\delta_{i1} + \frac{\partial \phi'}{\partial x_i}\right), \tag{12}$$

where $\frac{d\Phi}{dx_1}$ is associated with the difference of the mean pressure gradients from Eq. (9):

$$\frac{d\Phi}{dx_1} = \frac{dP^k}{dx_1} - \frac{dP^{k-1}}{dx_1}. \tag{13}$$

Because $\frac{dP^k}{dx_1}$ is *a priori* unknown, $\frac{d\Phi}{dx_1}$ cannot be directly obtained from Eq. (13). In order to obtain $\frac{dP^k}{dx_1}$ in terms of $\hat{u}_i$, u_i^{k-1} and p^{k-1}, we integrate Eq. (3) for $i=1$ under the constraint of a constant mass flow rate:

$$2\beta_k V\frac{dP^k}{dx_1} = \int [\beta_k(B_1^k + B_1^{k-1}) + \gamma_k A_1^{k-1} + \zeta_k A_1^{k-2}]\, dV. \tag{14}$$

From Eq. (6), Eq. (14) becomes

$$2\beta_k V\frac{dP^k}{dx_1} = \int \left[\frac{\hat{u}_1 - u_1^{k-1}}{\Delta t} + \beta_k(B_1^k - \hat{B}_1)\right] dV + 2\beta_k V\frac{dP^{k-1}}{dx_1}. \tag{15}$$

Then from Eqs. (13) and (15), we obtain

$$2\beta_k V\frac{d\Phi}{dx_1} = \int \left[\frac{\hat{u}_1 - u_1^{k-1}}{\Delta t} + \beta_k(B_1^k - \hat{B}_1)\right] dV. \tag{16}$$

Now, $\frac{d\Phi}{dx_1}$ can be easily obtained by applying Eq. (12) to B_1^k in Eq. (16) and rearranging the resulting equation.

Finally, the system of equations to be solved, which guarantees a constant mass flow rate in a fully developed channel or pipe, is as follows:

$$\hat{u}_i - \Delta t \beta_k \hat{B}_i = u_i^{k-1} + \Delta t(\beta_k B_i^{k-1} + \gamma_k A_i^{k-1} + \zeta_k A_i^{k-2}) - 2\beta_k \Delta t\left(\frac{dP^{k-1}}{dx_1}\delta_{i1} + \frac{\partial p'^{k-1}}{\partial x_i}\right), \tag{17}$$

$$\frac{\partial^2 \phi'}{\partial x_i \partial x_i} = \frac{1}{2\beta_k \Delta t}\frac{\partial \hat{u}_i}{\partial x_i}, \tag{18}$$

$$u_i^k = \hat{u}_i - 2\beta_k \Delta t\left(\frac{d\Phi}{dx_1}\delta_{i1} + \frac{\partial \phi'}{\partial x_i}\right). \tag{19}$$

The solution procedure is as follows. First solve Eq. (17) for the intermediate velocity, $\hat{u}_i$. After solving Eqs. (16) and (18) to obtain $\frac{d\Phi}{dx_1}$ and ϕ', the intermediate velocity field is projected onto a divergence-free velocity field using Eq. (19). Last, the pressure p^k is updated from Eq. (9).

3. Numerical Tests

The modified fractional step method suggested in Sec. 2 is tested for both laminar and turbulent flows in a channel or pipe. In the case of laminar flow, the initial velocity is provided to be uniform across the channel or pipe. Then, the velocity profile develops to a parabolic profile in time. In the case of turbulent flow, random disturbances are imposed on the parabolic velocity profile as an initial velocity field, and direct numerical simulations are performed.

The system of equations is solved on a staggered grid and the spatial discretization scheme is based on a second-order finite volume formulation.

3.1 Spatially periodic channel flows

In the simulation of channel flows, the convection terms are treated explicitly (A_i) and the diffusion terms are treated implicitly (B_i).

In the case of laminar flow, the Reynolds

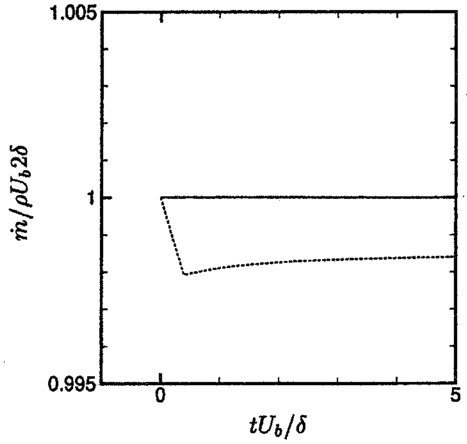

Fig. 1 Time history of the mass flow rate for laminar channel flow: ———, modified method; ·········, original method

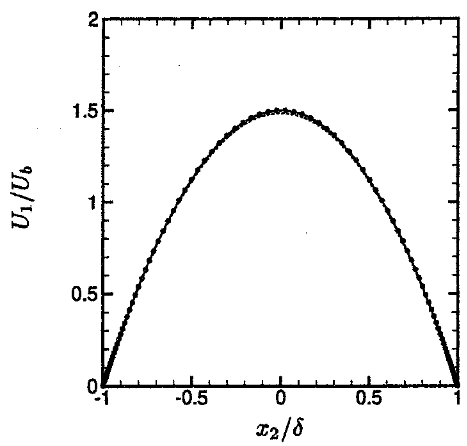

Fig. 2 Fully developed velocity profile for laminar channel flow: ———, modified method; ·········, original method; ●, analytical solution

number based on the bulk velocity, U_b, and the channel width, 2δ, is 1000. The present computations are carried out with 65 grid points equally spaced in the wall-normal (x_2) direction. Only one grid point is used in the streamwise (x_1) and spanwise (x_3) directions due to the periodic boundary condition. The mass flow rate per unit spanwise length is $\dot{m}/\rho U_b 2\delta = 1$. Figure 1 shows the time history of the mass flow rate inside the channel. At $t=0$, the velocity profile is given to be uniform. It is clear that the modified method keeps the mass flow rate constant in time, while

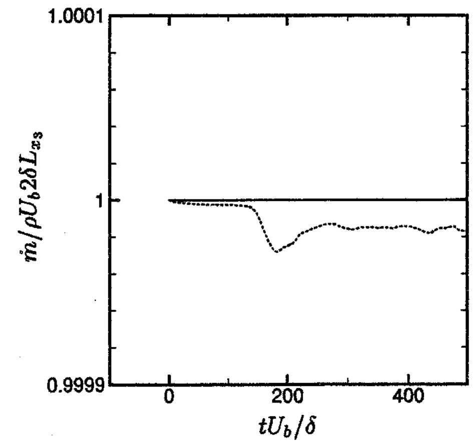

Fig. 3 Time history of the mass flow rate for turbulent channel flow: ———, modified method; ·········, original method

the original method does not. Figure 2 shows the fully developed velocity profile in the channel. With the original method, the velocity profile is not exact due to the decreased mass flow rate (Fig. 1).

Figure 3 shows the time history of the mass flow rate for turbulent channel flow. The computation is carried out at the Reynolds number of 4000 based on U_b and 2δ. This Reynolds number corresponds to $Re_\tau = 135$ based on the wall-shear velocity, u_τ, and the channel half width, δ. The computational domain sizes in the streamwise (x_1), wall-normal (x_2), and spanwise (x_3) directions are $L_{x_1} = 7\delta$, $L_{x_2} = 2\delta$, and $L_{x_3} = 3.5\delta$, respectively. The grid points used are $64 \times 97 \times 96$ in the x_1, x_2, and x_3 directions, respectively. Random disturbances are initially imposed on the laminar velocity profile. As is shown in Fig. 3, the mass flow rate is constant in time with the modified method, while the original method does not precisely guarantee a constant mass flow rate in time.

3.2 Spatially periodic pipe flows

In a pipe flow, the domain decomposition method is used to avoid severe time step limitations (Akselvoll & Moin, 1996). That is, in the core region all terms (convection and diffusion) with derivatives in the azimuthal direction are treated implicitly (B_i) and the remaining terms

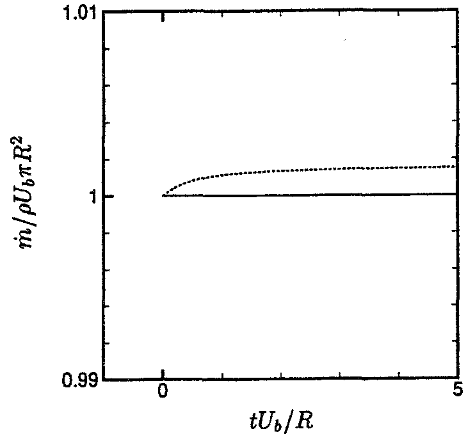

Fig. 4 Time history of the mass flow rate for laminar pipe flow: ———, modified method; ·········, original method

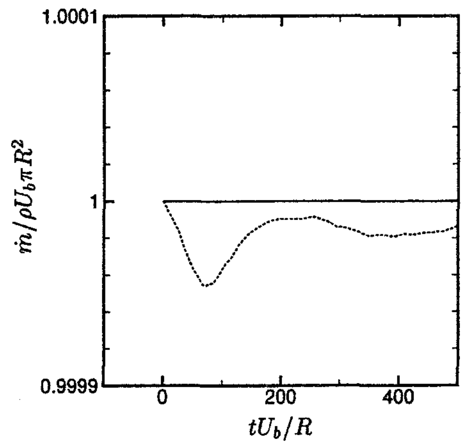

Fig. 5 Time history of the mass flow rate for turbulent pipe flow: ———, modified method; ·········, original method

explicitly (A_i), whereas in the outer region, convection and diffusion terms with derivatives in the radial direction only are treated implicitly.

Figure 4 shows the time history of the mass flow rate for laminar pipe flow. The Reynolds number is 1000 based on the bulk velocity, U_b, and the pipe diameter, $2R$. The present computations are carried out with 65×16 grid points equally spaced in the radial (x_2) and azimuthal (x_3) direction. Only one grid point is used in the streamwise (x_1) direction due to the periodic boundary condition. At $t=0$, a uniform velocity inside a pipe is provided. In this case, the mass

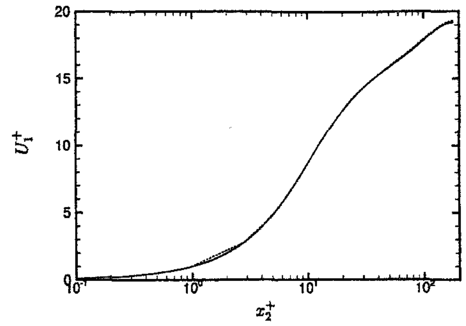

Fig. 6 Axial mean velocity profile in wall coordinates: ———, present result; ·········, DNS result of Eggels *et al.* (1994)

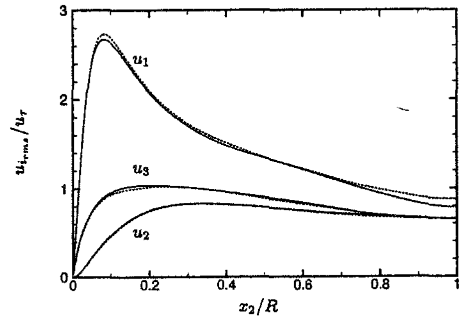

Fig. 7 Root-mean-square velocity fluctuations: ———, present result; ·········, DNS result of Eggels *et al.* (1994)

flow rate is $\dot{m}/\rho U_b \pi R^2 = 1$. It is clear that the modified method keeps the mass flow rate constant in time, while the original method does not.

Figure 5 shows the time history of the mass flow rate for turbulent pipe flow. The Reynolds number investigated is $Re = U_b \cdot 2R/\nu = 5300$, which corresponds to $Re_\tau = 180$ based on the wall-shear velocity, u_τ, and the radius, R. The computational domain consists of a pipe with the radius R and the streamwise length $10R$. The mesh contains $256 \times 69 \times 128$ grids in the axial (x_1), radial (x_2), and azimuthal (x_3) directions, respectively. It is shown that the mass flow rate is kept constant by the modified method. Figures 6 and 7 show the axial mean velocity profile and turbulence intensities, respectively, together with the DNS results of Eggels *et al.* (1994). The computational domain used in Eggels *et al.* (1994) was the same as that used in the present study. Eggels *et al.* (1994), however, treated all terms with derivatives in the azimuthal direction implicitly and others explicitly with a fixed mean pressure gradient in time. An excellent agreement is observed between the present result and DNS result of Eggels *et al.* (1994). This agreement guarantees that the modified fractional method can be successfully applied to pipe flow with the domain decomposition method.

4. Conclusion

In this paper, a modified fractional step method of keeping a constant mass flow rate in spatially periodic flows was presented, because original fractional step methods did not precisely keep the mass flow rate constant in time. In the modified method, the mean and fluctuating pseudo-pressure gradients were separately obtained at each time step. The modified fractional method was applied for both laminar and turbulent flows in a channel or pipe. It was shown that the modified fractional method kept the mass flow rate constant in time, whereas original fractional step methods did not.

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